CONFINEMENT AND CHIRAL DYNAMICS IN THE MULTI-FLAVOR SCHWINGER MODEL[a]

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Two-dimensional QED with N flavor fermions is solved at zero and finite temperature with arbitrary fermion masses to explore QCD physics such as chiral condensate and string tension. The problem is reduced to solving a Schrödinger equation for N degrees of freedom with a specific potential determined by the ground state of the Schrödinger problem itself.

1 QCD$_4$ physics in QED$_2$

Two-dimensional QED with massive fermions is not exactly solvable, ie. it is not reduced to a free theory like the massless case. When the fermion masses are large it becomes a highly interacting model. Due to its many similarities with four-dimensional QCD in such respects as instantons, chiral dynamics, and confinement, it has long been a testing ground for intuitive physical principles and new ideas.

One of the purposes of this work is to clarify these aspects of QCD physics by evaluating chiral condensates, the Polyakov loop, and string tension as fermion masses (m), vacuum angle (θ), and temperature (T) vary. We shall show that N flavor QED is effectively reduced to the quantum mechanics of N degrees of freedom, which can be solved numerically on workstations.[1,2]

The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \sum_{a=1}^{N} \overline{\psi}_a \Big\{ \gamma^\mu(i\partial_\mu - eA_\mu) - m_a \Big\} \psi_a \ . \tag{1}$$

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We examine the model defined on a circle S^1 with a circumference L. Upon imposing periodic and anti-periodic boundary conditions on the bosonic and fermionic fields, respectively, the model is mathematically equivalent to a theory defined on a line (R^1) at finite temperature (T) by analytic continuation to imaginary time τ $(= it)$ and interchange of τ and x. Various physical quantities at $T \neq 0$ on R^1 are obtained from the corresponding ones at $T = 0$ on S^1 by substituting T^{-1} for L.

2   Zero-mode Hamiltonian

To begin the computation we bosonize the fermions. Each two-component fermion (ψ_a) is expressed in terms of zero modes $(q_a^\pm, p_a^\pm)$ and oscillatory modes $(\phi_a(x), \Pi_a(x))$. The boundary conditions enforce that $p_a^\pm$'s take integer eigenvalues. $p_a^- - p_a^+$ and $p_a^- + p_a^+$ correspond to charge and chiral charge, respectively. In a vector-like theory without background charges one can stay in a subspace defined by $p_a^+ = p_a^-$.

The relevant parts of the Hamiltonian are expressed in terms of $(q_a{=}q_a^+ + q_a^-, p_a{=}\frac{1}{2}[p_a^+ + p_a^-])$, (ϕ_a, Π_a), and (Θ_W, P_W) where Θ_W is the Wilson line phase, the only physical degree of freedom associated with gauge fields on a circle. The Hamiltonian becomes

$$H_{\text{tot}} = H_0 + H_\phi + H_{\text{mass}}$$

$$H_0 = \frac{\pi \mu^2 L}{2N} P_W^2 + \frac{1}{2\pi L} \sum_{a=1}^{N} (\Theta_W + 2\pi p_a)^2$$

$$H_\phi = \int_0^L dx \, \frac{1}{2} \left\{ \sum_{a=1}^{N} \left(\Pi_a{}^2 + \phi_a'^2 \right) + \mu^2 \left(\frac{1}{\sqrt{N}} \sum_a \phi_a \right)^2 \right\} \tag{2}$$

where $\mu^2 = Ne^2/\pi$. H_{mass} represents the contribution coming from the fermion mass terms. Expression (2) is exact, from which it immediately follows that QED$_2$ with massless fermions is exactly solvable. It contains 1 massive boson and $N - 1$ massless bosons.

The fermion masses give nontrivial interactions among the zero modes and ϕ modes, and the previously massless bosons now become massive. To find the true vacuum, we first determine the vacuum wave function in the zero mode sector with given physical boson masses μ_α's $(\alpha{=}1, \cdots, N)$. The boson masses, which depend on the vacuum structure in the zero mode sector, are recomputed with the vacuum wave function thus obtained. Since input and output values for the boson masses must be the same, this gives a self-consistency condition which we can solve for numerically in general, and analytically in certain limits.

As a basis one may take eigenstates of (P_W, p_a) with eigenvalues (p_W, n_a) where the n_a's are integers. It is more convenient however to take a coherent state basis with respect to $\{n_a\}$. The vacuum wave function is written as $\hat{f}(p_W; \varphi_1, \cdots, \varphi_{N-1}; \theta)$. One of the angular variables, θ, specifies the so-called θ-vacuum; due to gauge invariance its value does not change.

When fermion masses are small, $\hat{f} = e^{-\pi \mu L p_W^2 / 2N} f(\varphi; \theta)$ to good accuracy. $f(\varphi; \theta)$ must satisfy

$$\left\{ -\triangle_N + V_N \right\} f(\varphi_1, \cdots, \varphi_{N-1}) = \epsilon \, f(\varphi_1, \cdots, \varphi_{N-1})$$

$$\triangle_N = \sum_{a=1}^{N-1} \frac{\partial^2}{\partial \varphi_a^2} - \frac{2}{N-1} \sum_{a<b}^{N-1} \frac{\partial^2}{\partial \varphi_a \partial \varphi_b}$$

$$V_N = -\sum_{a=1}^{N} m_a A_a \cos \varphi_a \quad \left(\sum_{a=1}^{N} \varphi_a = \theta \right) \tag{3}$$

where A_a is determined by the boson masses μ_α. The problem is now a Schrödinger equation. The salient feature is that the potential has to be determined self-consistently such that its ground state wave function (and hence the boson masses μ_α) reproduces the same potential:

$$V_N(\varphi) \rightarrow f(\varphi) \rightarrow \mu_\alpha \rightarrow V_N(\varphi) . \tag{4}$$

3 Chiral dynamics

It is straightforward to determine the chiral condensate. In the large or small volume limit (or equivalently in the low or high temperatue limit) an analytic expression can be obtained. For $N \geq 3$ with degenerate fermion masses ($m_a = m \ll \mu$)

$$\frac{1}{\mu} \langle \overline{\psi} \psi \rangle_\theta = \begin{cases} -\dfrac{1}{4\pi} \left(2e^\gamma \cos \dfrac{\overline{\theta}}{N} \right)^{\frac{2N}{N+1}} \left(\dfrac{m}{\mu} \right)^{\frac{N-1}{N+1}} & \text{for } T \ll m^{\frac{N}{N+1}} \mu^{\frac{1}{N+1}} \\ -\dfrac{2N}{\pi(N-1)} \dfrac{m}{\mu} e^{-2\pi T / N \mu} & \text{for } T \gg \mu . \end{cases} \tag{5}$$

Here $\overline{\theta}$ is defined in the interval $-\pi \leq \overline{\theta} \leq +\pi$ by $\overline{\theta} = \theta - 2\pi[(\theta + \pi)/2\pi]$.

A few conclusions can be drawn here. First of all, at low T the chiral condensate is not analytic in m. This point was noticed in the $N = 2$ case at $T = 0$ by Coleman twenty years ago.[4] Secondly, at $T = 0$, a cusp singularity appears at $\theta = \pi$. Thirdly at high temperature perturbation theory in the fermion masses is applicable since the condensate becomes analytic in m.
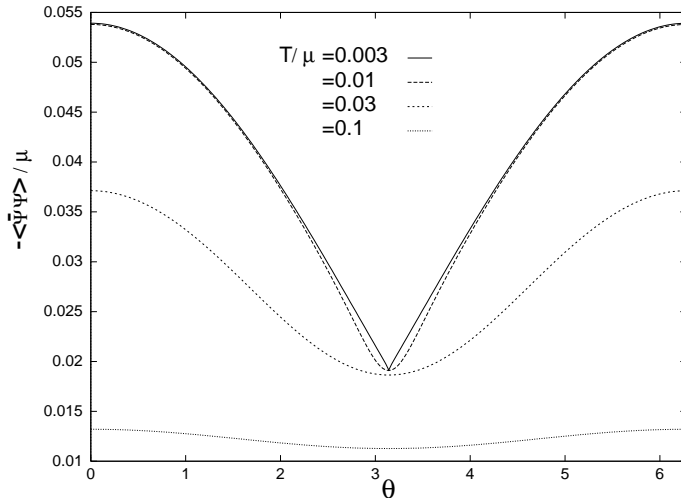
Figure 1: θ-dependence of the chiral condensate at various temperatures in the $N = 3$ model with $m/\mu = .01$. At $T = 0$, a cusp singularity appears at $\theta = \pi$.

At moderate temperatures Eq. (3) must be solved numerically. In fig. 1 we have displayed the θ dependence of chiral condensate in the $N = 3$ case with $m/\mu = 0.01$ at various temperatures. One can see the cusp singularity develop as T approaches zero.

At very low T (large L) the potential term dominates over the kinetic energy term in Eq. (3). In other words, the ground state wave function $f(\varphi)$ has a sharp peak around the location of the absolute minimum of the potential $V_N(\varphi)$. As θ varies, the location of the minimum also changes. In the case of degenerate fermion masses, the minimum is located at $\varphi_a = \bar{\theta}/N$. The location of the minimum discontinuously shifts at $\theta = \pi$ (mod 2π), which is the origin of the cusp singularity encountered in the θ dependence of the chiral condensate at $T = 0$.

When the fermion masses are not degenerate, the coefficients $m_a A_a$ are all distinct. In the three flavor ($N = 3$) case the potential takes the form

$$V_3[\varphi] = -\left\{q_1 \cos \varphi_1 + q_2 \cos \varphi_2 + q_3 \cos(\theta - \varphi_1 - \varphi_2)\right\} \tag{6}$$

where all q_a's are different. This potential has the same structure as the effective chiral Lagrangian in QCD. In the effective Lagrangian written by

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Witten [5] a potential term reads

$$V^{\text{Witten}}(U) = f_\pi^2 \left\{ -\frac{1}{2} \text{Tr}\, M(U + U^\dagger) + \frac{k}{2N_c}(-i \ln \det U - \theta)^2 \right\} \qquad (7)$$

Here U is the pseudoscalar field matrix, whereas $M = diag\,(m_u, m_d, m_s)$ is the quark mass matrix . The second term represents the contribution from instantons. The coefficient k is O(1) in the large N_{color} limit.

Phenomenologically $m_{\eta'}^2 \gg m_\pi^2, m_K^2, m_\eta^2$, which implies that $k/N_c \gg m_a$, or that upon diagonalizing $U = \text{diag}\,(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$, $\sum \phi_a = \theta$. Hence (7) reduces to (6). These equations are not exactly the same however, as q_a in (6) is not simply m_a but depends on $\{m_a\}$ rather nontrivially.

In the past various conclusions were drawn based on (7), which turn out to be perfectly correct in our context as well. Physical quantities are periodic in θ with period 2π. With degenerate quark masses a cusp singularity appears at $\theta = \pi$. Furthermore sufficiently large asymmetry in quark masses removes the singularity.[6]

Indeed, we have observed that at $T = 0$ the location of the minimum of the potential determines the vacuum wave function. When $q_1 = q_2 \ll q_3$, i.e. the strange quark is heavy but the up and down quarks are degenerate ($m_u = m_d \ll m_s$), we find that the discontinuous jump at $\theta = \pi$ in the location of the minimum remains. However, if one adds small asymmetry in the up and down quark masses ($m_u < m_d \ll m_s$), the minimum moves continuously to make a loop as θ changes from $-\pi$ to $+\pi$. In other words the cusp singularity disappears.

4 Confinement

One common method of examining confinement is to evaluate the Polyakov loop. In our method this is simply the expectation value of the Wilson line phase:

$$P_q = \left\langle e^{iq \int_0^\beta d\tau\, A_0(\tau, x)} \right\rangle = e^{-\beta F_q} \Leftrightarrow \left\langle e^{i(q/e)\Theta_W(t)} \right\rangle_{L=T^{-1}}$$

$$= \begin{cases} 0 & \text{for } \dfrac{q}{e} \neq \text{integer} \\[2ex] e^{-\frac{\pi \mu n^2}{4NT}} \int [d\varphi]\, f(\varphi_a)^* f(\varphi_a + \frac{2\pi n}{N}) & \text{for } \dfrac{q}{e} = n \end{cases} \qquad (8)$$

The result for $q = e$ is displayed in fig. 2. Although P_q shows a crossover transition and becomes vanishingly small at low T, the free energy F_q of a test
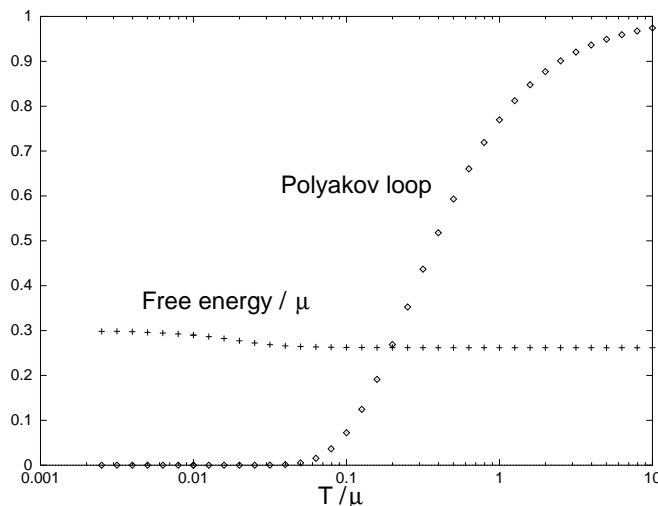
Figure 2: T-dependence of the Polyakov loop and free energy in the $N = 3$ model with $m/\mu = .01$. The free energy stays finite at all T.

charge q remains finite at all T. P_q vanishes for a fractional charge. However, one cannot conclude that confinement of fractional charge from this result alone, as the vanishing of P_q follows solely from gauge invariance.

To get more information about confinement, one should insert a pair of sources, one with charge q and the other with $-q$, and examine the increase (or decrease) of the energy. The shift in the energy is parametrised as $\Delta E = \sigma d + \cdots$ where d is the distance between the two sources, and σ is the string tension.

In the multi-flavor case perturbation theory in mass cannot be employed. Nevertheless one arrives at a simple result. External charges are completely screened, but the effective θ value is shifted between the two sources by an amount $2\pi q/e$. This shift in turn changes the chiral condensate and therefore the energy density between the sources[3,7,8,9]

One finds

$$\sigma = Nm \left\{ \langle \bar{\psi}\psi \rangle_{\theta_{\text{eff}}} - \langle \bar{\psi}\psi \rangle_{\theta} \right\} \quad , \quad \theta_{\text{eff}} = \theta - \frac{2\pi q}{e} \ . \tag{9}$$

In particular, at $T = 0$

$$\frac{\sigma}{\mu^2} = -\frac{N}{2\pi} \left(2e^{\gamma} \frac{m}{\mu} \right)^{\frac{2N}{N+1}} \left\{ \left(\cos \frac{\bar{\theta}_{\text{eff}}}{N} \right)^{\frac{2N}{N+1}} - \left(\cos \frac{\bar{\theta}}{N} \right)^{\frac{2N}{N+1}} \right\} \tag{10}$$

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Notice that the string tension vanishes for an integer q/e. The string tension σ is non-vanishing only when fermions are massive and chiral condensates have non-trivial θ dependence.

5 Heavy fermions

When fermion masses become large, we need to solve a more general problem. For one flavor there is no φ degree of freedom. The vacuum wave function is expressed as $\hat{f}(p_W)$ which must satisfy

$$\left\{ -\frac{d^2}{dp_W^2} + \omega^2\, p_W^2 - k\cos(2\pi p_W + \theta) \right\} \hat{f}(p_W) = \epsilon \hat{f}(p_W) \ . \qquad (11)$$

Here $\omega = \pi\mu L$ and k depends on the fermion mass and the wave function itself. One finds that

$$\langle \overline{\psi}\,\psi \,\rangle_{T=0} \sim \begin{cases} -\dfrac{\mu}{2\pi}\, e^{\gamma}\cos\theta & \text{for } m \ll \mu \\[2mm] -\dfrac{m}{\pi}\, e^{2\gamma} & \text{for } m \gg \mu \ . \end{cases} \qquad (12)$$

The chiral condensate becomes bigger as the fermion mass m grows. The effect of a very heavy fermion never disappears even if m approaches infinity. Does this contradict the decoupling theorem, that heavy fermions are irrelevant in low energy physics?

The answer is 'no'. Chiral condensates of heavy fields are not good measures of low energy physics. Instead one should examine, for instance, the string tension defined when a pair of external sources is inserted as was done in the previous section.

Eq. (9) shows that the string tension σ depends on two factors: m and the θ dependence of $\langle \overline{\psi}\,\psi \,\rangle$. Although $m\langle \overline{\psi}\,\psi \,\rangle$ becomes very large as m increases, its θ dependence diminishes rapidly for a large m. In fig. 3 we have displayed the m dependence of σ for $\theta = 0$ and $q/e = \frac{1}{2}$. One can see that σ increases for moderate m, but quickly approaches zero as m becomes large. Extremely heavy fermions are thus irrelevant for low energy physics.

6 Summary

There are similarities and differences between QCD_4 and QED_2: both have confinement, and their chiral dynamics are pretty much the same. However, in QED_2 a non-vanishing string tension (ie. confinement) results only if $m\langle \overline{\psi}\,\psi \,\rangle$
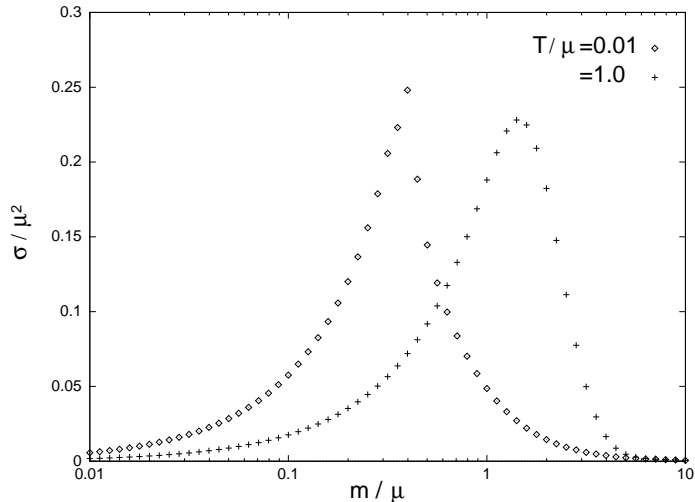
Figure 3: Mass dependence of the string tension σ in the $N = 3$ model. σ vanishes for a large fermion mass, which is consistent with the decoupling theorem.

has non-trivial θ dependence. In other words, if there were no chiral condensates, there would be no confinement in QED_2.

This can hardly be true in QCD_4 where it seems that confinement and spontaneous chiral symmetry breaking are two separate phenomena. Chiral symmetry is spontaneously broken even in the chiral limit $m_a = 0$, which is not the case in multiflavor QED_2. Nonetheless lattice simulations show that confinement and chiral symmetry breaking are intimately related.[10]

Despite these differences, we can still learn quite a bit of QCD physics from QED_2. After all we have determined various physical quantities such as chiral condensates, Polyakov loop, and string tension at any temperature and with arbitrary fermion masses. This is the luxury of two-dimensional gauge theory. In the work summarized here, we have presented a powerful method for solving QED_2 which compliments lattice gauge theory and light-front methods.[11]

Acknowledgments

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